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Pure $SU(2)$ Gauge Field

H. Dehnen[1] and F. Ghaboussi[1]

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The general structure of the pure $SU(2)$ gauge potentials is calculated in detail. It is shown that the expansion of the gauge potentials about nonvanishing pure gauge potentials gives rise to new effects with confinement character.

In the modern context of quantum field theory one tries to explain all interactions and their unification by the unitary gauge principle. Following this line, we have proposed a Yang-Mills gauge theory for gravity based on the group $U(2)$ of the two-spinor transformations (Dehnen and Ghaboussi, 1986); in the classical limit the metric tensor of gravity according to Einstein's theory is built up by products of the Yang-Mills gauge potentials (Ghaboussi *et al.*, 1987). In this connection the investigation of pure Yang-Mills gauge structures is of general importance.

Usually in gauge theories the gauge potentials are chosen to be zero if the gauge field strength vanishes identically. However, gravity represents a significant exception: The nonvanishing Minkowski metric plays the role of the pure gauge field of classical gravity, for which the Riemann tensor (field strength of classical gravity) vanishes; the connections (Christoffel symbols) are zero in this case only with respect to Cartesian space-time coordinates. Then, real gravity is described by the deviation of the metric tensor from the Minkowski metric.

A similar situation is of interest in the case of Yang-Mills gauge theories, for instance, in view of our proposal of a Yang-Mills gauge theory of gravity, where in a consistent manner the Minkowski metric must be produced by the product of nonvanishing pure Yang-Mills gauge potentials. The real gauge potentials corresponding to real gravity are given then by the deviation from the nonvanishing pure gauge potentials (Dehnen and Ghaboussi, 1987). Of course, this procedure is connected with a certain gauge-fixing.

[1]Fakultät für Physik der Universität Konstanz, D7750 Konstanz, W^est Germany.

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For this reason we calculate in the following the general structure of the pure gauge potentials in the case of the group $SU(2)$. These are defined by (g is the coupling constant)

$$\overset{(0)}{\omega}_{\mu}=(i/g)U_{|\mu}U^{-1} \tag{1}$$

with the unitary matrix

$$U=\exp[i\lambda_a(x^\nu)\tau^a], \qquad \tau^a=\tfrac{1}{2}\sigma^a \tag{1a}$$

(σ^a are Pauli matrices; λ_a are real gauge functions). This representation is identical with the condition of vanishing field strength:

$$F_{\mu\nu}=(1/ig)[D_\mu, D_\nu]=0 \tag{2}$$

where $D_\mu=\partial_\mu+ig\omega_\mu$ with $\omega_\mu=\overset{(0)}{\omega}_{\mu}$.

Now we are interested in the explicit structure of the potentials $\omega_{\mu a}$ given by

$$\omega_\mu=\omega_{\mu a}\tau^a \tag{3}$$

which satisfy the condition (2). By insertion of (1a) into (1) and by subsequent expansion of the exponential function in (1a) one finds, using the algebra of τ^a, the following result after a long, but simple calculation:

$$\overset{(0)}{\omega}_{\mu a}=-(1/g)(\lambda_{a|\mu}\cdot f-\lambda_b\lambda_{c|\mu}\varepsilon_a{}^{bc}\cdot p+\lambda_a\lambda^b\lambda_{b|\mu}\cdot k) \tag{4}$$

Here $\varepsilon_a{}^{bc}$ are the structure constants of $SU(2)$, and the functions f, k, and p are given by

$$\begin{aligned} f&=(\sin\lambda)/\lambda, \qquad p=(1-\cos\lambda)/\lambda^2 \\ k&=(\lambda-\sin\lambda)/\lambda^3, \qquad \lambda^2=\lambda^a\lambda_a \end{aligned} \tag{4a}$$

For small values of λ_a the relation (4) goes over into three pure $U(1)$ gauge potentials

$$\overset{(0)}{\omega}_{\mu a}=-(1/g)\lambda_{a|\mu}$$

One proves easily that, with (4) and (4a), the condition (2) is satisfied. From

$$F_{\mu\nu}=F_{\mu\nu a}\tau^a \tag{5}$$

with

$$F_{\mu\nu a}=\omega_{\nu a|\mu}-\omega_{\mu a|\nu}-g\varepsilon_a{}^{bc}\omega_{\mu b}\omega_{\nu c}$$

it follows with

$$\omega_{\mu a} = \overset{(0)}{\omega}_{\mu a}$$

that, according to (4),

$$\overset{(0)}{F}_{\mu\nu a} = (1/g)\left\{\left(\lambda_{a|\mu}\lambda^{b}\lambda_{b|\nu} - \lambda_{a|\nu}\lambda^{b}\lambda_{b|\mu}\right)\left(\frac{1}{\lambda}f' - k + fp + \lambda^2 kp\right)\right.$$
$$+ \varepsilon_a{}^{bc}\left[\lambda_{b|\mu}\lambda_{c|\nu}(2p - f^2 - \lambda^2 p^2)\right.$$
$$\left.\left. + (\lambda_{b|\mu}\lambda_c\lambda^{d}\lambda_{d|\nu} + \lambda_b\lambda_{c|\nu}\lambda^{d}\lambda_{d|\mu})\left(\frac{1}{\lambda}p' - fk + p^2\right)\right]\right\} \tag{6}$$

where the prime signifies $\partial/\partial\lambda$, and

$$\overset{(0)}{F}_{\mu\nu a} = F_{\mu\nu a}(\overset{(0)}{\omega}_{\sigma b})$$

With the use of (4a) one gets immediately

$$\begin{aligned} (1/\lambda)f' - k + fp + \lambda^2 kp &= 0 \\ 2p - f^2 - \lambda^2 p^2 &= 0 \\ (1/\lambda)p' - fk + p^2 &= 0 \end{aligned} \tag{6a}$$

so that with respect to (6)

$$\overset{(0)}{F}_{\mu\nu a} = 0 \Leftrightarrow \overset{(0)}{F}_{\mu\nu} = 0 \tag{7}$$

is valid.

Finally, we point to the effect of the expansion of the gauge potentials $\omega_{\mu a}$ around the nonvanishing pure gauge potentials, as mentioned above:

$$\omega_{\mu a} = \overset{(0)}{\omega}_{\mu a} + A_{\mu a} \tag{8}$$

The field strength (5) takes the form, using (7),

$$F_{\mu\nu a} = A_{\nu a|\mu} - A_{\mu a|\nu} - g\varepsilon_a{}^{bc}(\overset{(0)}{\omega}_{\mu b}A_{\nu c} + \overset{(0)}{\omega}_{\nu c}A_{\mu b} + A_{\mu b}A_{\nu c}) \tag{9}$$

Obviously we obtain in this way two additional terms in $F_{\mu\nu a}$ proportional to the real potentials $A_{\mu a}$. It is evident that these give rise to additional terms also in the field equations for $A_{\mu a}$, which is of general interest, because these terms are present even in the linearized version with respect to $A_{\mu a}$. As a consequence, a confinement structure for the potentials $A_{\mu a}$ can be expected. Elsewhere we have shown (Dehnen and Ghaboussi, 1987) that with an ansatz corresponding to (8) the following differential equation for the static $SU(2)$ potentials A_{0a} results (linearized in $A_{\mu a}$):

$$A_{0a}{}^{|m}{}_{|m} - 2g\varepsilon_a{}^{mn}A_{0n|m} - 2g^2 A_{0a} = 0 \tag{10}$$

Besides the solution discussed in Dehnen and Ghaboussi (1987), this equation still possesses a second spherically symmetric, but asympotically nonvanishing solution:

$$A_{0a} \sim \partial_a[r^{-1} \exp(2^{1/2}gr) \tag{11}$$

Then the static field strength following from (9) and (11) is given by

$$F_{m0a} = \partial_m \partial_a[r^{-1} \exp(2^{1/2}gr)] - g\varepsilon_{am}{}^n \partial_n[r^{-1} \exp(2^{1/2}gr)] \tag{12}$$

In this case a confinement behavior exists for $r \gtrsim 1/2^{1/2}g$. A detailed investigation in this direction is in preparation.

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